Stiffness Modeling for Multi-Fingered Grasping with Rolling Contacts

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***Abstract*— The stiffness control of an object grasped by a multi-fingered robot hand requires the modeling of the elastic behavior of the object, caused by the stiffness of the fingers. Because of the presence of rolling contacts between the fingers and the object, such a modeling is not a trivial issue, and a very different one from the case of simpler parallel manipulators. We provide here a first expression of the cartesian stiffness matrix produced on the object, as a function of the cartesian stiffness matrices of the fingers, in the case that the contacts are non-sliding point contacts that may freely roll (on the tangent plane) and twist (around the contact normal). We show that this expression of the object-level cartesian stiffness matrix depends also on the contact forces and on the local geometries of the contacting surfaces.**

***Keywords*— Multi-fingered hand, stiffness control, stiffness matrix, rolling contacts.**

I. Introduction

A. Problem statement

1) Stiffnesses at joint level and phalanx level: In a robotic multi-fingered hand, each joint of each finger may have a certain, characteristic stiffness, termed passive when it arises from structural reasons and active when it results from motor control. These articular stiffnesses may be put together in a characteristic joint stiffness matrix for each finger. Let $K_{art,i}$ denote this stiffness matrix for finger i. It is a square $(n_{dof,i}, n_{dof,i})$ matrix, with $n_{dof,i}$ the number of degrees of freedom of finger i. Its diagonal terms are the stiffnesses of the joints and its off-diagonal terms, if any, are stiffness couplings between the joints.

This articular stiffness of finger i results in an equivalent $(6, 6)$ cartesian-space stiffness, at distal phalanx level (i.e. at end-effector level). We let K_{dp_i} denote this cartesian stiffness matrix for finger i, at phalanx level, written in the phalanx body-fixed frame.

The relation between $K_{art,i}$ and K_{dp_i} is not exactly a standard change of frame formula between the joint and cartesian spaces, contrary to what was commonly admitted since the early works of Salisbury [1]. That is to say, $K_{art,i} \neq J_i^T K_{dp_i} J_i$, with J_i the jacobian matrix of finger i. Rather, an additional term must be taken into account, as explained twenty years later by Chen and Kao [2], [3]: $K_{art,i} - K_{g,i} = J_i^T K_{dp_i} J_i$. This is because the changes in the geometry of the finger as it moves under the effect of the contact force between the end-effector and the object, as well as the value of this contact force, both play a part in the resulting stiffness K_{dp_i}. The additional term $K_{g,i}$ describes their contribution to the resulting stiffness at phalanx level.

2) Resulting stiffness at object level: When the robot hand grasps an object, the finger stiffnesses induce a total resulting cartesian-space stiffness at object level. We let K_{obj} denote the $(6, 6)$ cartesian stiffness matrix, at object level, written in the object body-fixed frame.

It is clear that if the contacts were fixtures, then this resulting object stiffness would just be the sum of the cartesian stiffnesses K_{dp_i}, modulo changes of frame between the object and the distal phalanxes. See, for instance, [4, equation 7], where the stiffnesses to add describe simple, one-dimensional springs, or [5, end of section 3], where they are more general.

However, fingertips are generaly not fixed on the object. If they are round, they may roll on the surface of the object, and they probably *will* roll in the course of most motions. Fingertips may slide too, although we will assume in this work that it is not the case.

Contrary to Chen and Kao's relation between stiffnesses at joint level and end-effector level [2], [3], the relation between the finger stiffnesses and the resulting object-level stiffness is not yet known, in the case that the contacts are not fixtures. Yet such knowledge may be of valuable interest, as it would for instance enable the design of a multi-fingered robot hand, or of its stiffness control, in order to realize a certain desired stiffness on the grasped object. So far, we are unaware of studies that have adressed this problem.

B. Contribution

This paper provides the first insights into the modeling of the total stiffness resulting at object level from the stiffness of the fingers in the grasp.

Its first contribution is to prove that in a stiffness analysis, the infinitesimal pose variations of the object and a distal phalanx are linearly dependent one another. We provide an expression of this linear map that shows that it depends on the cartesian stiffness of the finger, on the contact force and on the local geometries of the contacting surfaces.

Its second contribution is to formulate an expression of the resulting stiffness K_{obj} as a function of the finger cartesian stiffnesses K_{dp_i}. This expression also depends on the contact forces and on the local geometries of the contacting surfaces. Unfortunately, it is far from the elegant simplicity of the congruence transformation that relates the joint stiffness and cartesian stiffness of only one finger.

These two results are valid under the following model hypotheses: all the bodies are rigid bodies, the contacts are non-sliding point contacts with friction, rolling and twisting of the contacts are possible, and they never break. We will also need to assume the invertibility of a certain matrix.

C. Outline of the paper

The rest of this paper is as follows. We define our notations and our model in section II; its hypotheses are clearly stated and translated into our notations. We introduce in section III the various equations needed by our modeling. Section IV demonstrates the linear map between the infinitesimal pose variations of the object and a distal phalanx, and section V formulates the relation we are looking for between K_{obj} and K_{dp_i}. Section VI gives numerical results in simulation, and section VII concludes the paper.

II. Model and notations

A. Rigid body mechanics: twists and wrenches

We explain briefly a few notations from rigid body mechanics. First, we let $V^a_{S_2/S_1}$ denote the twist, i.e. the generalized velocity, of some rigid body S_2 relatively to some other rigid body S_1, written in some frame a. We also let $W^a_{S_1\to S_2}$ denote the wrench, i.e. the generalized force, applied by the rigid body S_1 to the rigid body S_2, written in the frame a:

$$V^a_{S_2/S_1} = \begin{pmatrix} v^a_{A\in S_2/S_1} \\ \omega^a_{S_2/S_1} \end{pmatrix} \qquad W^a_{S_1\to S_2} = \begin{pmatrix} f^a_{S_1\to S_2} \\ m^a_{A,S_1\to S_2} \end{pmatrix}$$

In these expressions, A is the origin of the frame a, so that $m^a_{A,S_1\to S_2}$ is the moment in A applied by S_1 to S_2, written in the basis a, and $v^a_{A\in S_2/S_1}$ is the velocity of A, *considered as a fixed point* of S_2, relatively to S_1, written in the basis a. The other components, $f^a_{S_1\to S_2}$ and $\omega^a_{S_2/S_1}$, are respectively the force applied by S_1 to S_2 and the rotational velocity of S_2 relatively to S_1, both written in the basis a; they do not depend on the point at which the twist or wrench is written. When writing twists, we often omit S_1 if it is the reference body, the "world", i.e. for absolute twists: $V^a_{S_2} = V^a_{S_2/ref}$.

In our notations, the frame or basis specified at top-right position is the frame or basis in which the quantity is written, whatever the quantity. To write a twist or a wrench in another frame, we use the following change of frame formulas:

$$V^a_{S_2/S_1} = {}^aAd_b V^b_{S_2/S_1} \qquad W^a_{S_1\to S_2} = {}^aAd_b^{-T} W^b_{S_1\to S_2}$$

$${}^aAd_b = \begin{pmatrix} {}^aR_b & \hat{r}^a_{a,b}\,{}^aR_b \\ 0_{3,3} & {}^aR_b \end{pmatrix} \qquad {}^aAd_b^{-T} = \begin{pmatrix} {}^aR_b & 0_{3,3} \\ \hat{r}^a_{a,b}\,{}^aR_b & {}^aR_b \end{pmatrix}$$

${}^aAd_b$ and ${}^aAd_b^{-T}$ are called respectively adjoint and co-adjoint matrices (of the rigid body transformation from frame a to frame b). ${}^aR_b$ is the rotation matrix of basis b with respect to basis a, $r^a_{a,b} = \overrightarrow{AB}^a$ is the vector between the origins of the frames, written in basis a, and $\hat{r}^a_{a,b}$ is the following skew-symmetric matrix, embedding the operation of left-wise cross-product by vector $r_{a,b}$, in a coordinates:

$$r^a_{a,b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mapsto \quad \hat{r}^a_{a,b} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

this matrix meets: $\forall\, u \in \mathbb{R}^3,\ \hat{r}^a_{a,b} u^a = r^a_{a,b} \times u^a$

We let Π denote a matrix that selects the first component of a twist or a wrench, for instance $v^a_{A\in S_2/S_1} = \Pi V^a_{S_2/S_1}$, and Π' denote the one that selects the other component, as in $m^a_{A,S_1\to S_2} = \Pi' W^a_{S_1\to S_2}$:

$$\Pi = \begin{pmatrix} I_3 & 0_{3,3} \end{pmatrix} \qquad \Pi' = \begin{pmatrix} 0_{3,3} & I_3 \end{pmatrix}$$

Similarly to the cross-product matrix $\hat{r}^a_{a,b}$, we define the following two matrices, relative to a twist and a wrench, and by extension we also denote them with hats and refer to them as cross-product matrices:

$$\widehat{V}^a_{S_2/S_1} = \begin{pmatrix} \hat{\omega}^a_{S_2/S_1} & \hat{v}^a_{A\in S_2/S_1} \\ 0_{3,3} & \hat{\omega}^a_{S_2/S_1} \end{pmatrix}$$

$$\widehat{W}^a_{S_1\to S_2} = \begin{pmatrix} 0_{3,3} & \hat{f}^a_{S_1\to S_2} \\ \hat{f}^a_{S_1\to S_2} & \hat{m}^a_{A,S_1\to S_2} \end{pmatrix}$$

It is worth noting that the wrench-relative cross-product matrix is skew-symmetric: $(\widehat{W}^a_{S_1\to S_2})^T = -\widehat{W}^a_{S_1\to S_2}$.

We also formulate an infinitesimal displacement $\delta X^a_{S_2/S_1}$ of body S_2 relatively to body S_1, during dt, and written in frame a, as:

$$\delta X^a_{S_2/S_1} = V^a_{S_2/S_1} dt = \begin{pmatrix} \delta x^a_{A\in S_2/S_1} \\ \delta\theta^a_{S_2/S_1} \end{pmatrix}$$

The vector $\delta\theta_{S_2/S_1}$ is along the instantaneous axis of rotation of body S_2 relatively to body S_1.

In the rest of this paper, some quantities miss a frame specification in the top-right position, for brevity of the expressions. When unspecified, a frame is the most "natural" frame for the quantity. For instance, we have already encountered K_{dp_i} and K_{obj}, written respectively in the phalanx and object body-fixed frames, that is to say $K_{dp_i} = K^{dp_i}_{dp_i}$ and $K_{obj} = K^{obj}_{obj}$. Further, we will introduce the infinitesimal deflections in the finger contact forces and total contact force, and note them respectively $dW_{dp_i\to obj} = dW^{dp_i}_{dp_i\to obj}$ and $dW_{dp\to obj} = dW^{obj}_{dp\to obj}$.

B. Hand and object models

The robot grasp we consider consists of n_f hard-fingers grasping a rigid object in three-dimensional space at n_f point contacts with dry friction. Finger $i \in [|1, n_f|]$ is illustrated on figure 1. We place no restriction on the number of phalanxes and joints, and let $n_{dof,i}$ denote the number of degrees of freedom of finger i.

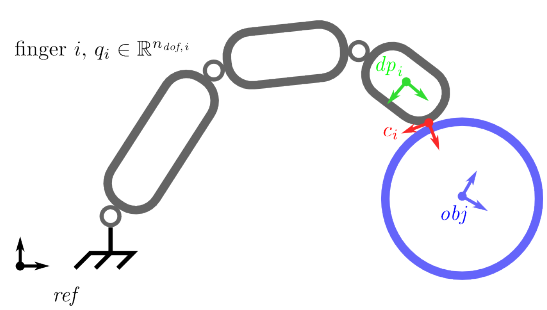


Fig. 1. Finger i, $i \in [|1, n_f|]$

We let $q_i \in \mathbb{R}^{n_{dof,i}}$ denote the articular configuration of finger i. dp_i denotes both the distal phalanx and its main frame, located at the phalanx center of mass. ref is an inertial reference frame and obj is the object, or its frame. c_i is both the contact point and a contact frame at the object/finger interface, with outward-pointing normal with respect to the distal phalanx.

C. Contact model hypotheses

In this paper, we assume that the point contacts are non-sliding and that rolling (on the tangent plane) and twisting (around the contact normal) are free. We also assume that the contacts always hold.

We note $V^{c_i}_{dp_i/obj}$ the twist of the relative motion between the phalanx and the object. We also note $v_{c_i \in dp_i/obj}$ and $\omega_{dp_i/obj}$ the translational and rotational velocities of this relative motion, velocities of which different components, in c_i coordinates, are commonly known as the *sliding*, *rolling*, *twisting* and *breaking* velocities between the phalanx and the object. Namely:

$$(v^{c_i}_{c_i \in dp_i/obj})_{x,y} = \text{sliding} \qquad (\omega^{c_i}_{dp_i/obj})_{x,y} = \text{rolling}$$
$$(v^{c_i}_{c_i \in dp_i/obj})_{z} = \text{breaking} \qquad (\omega^{c_i}_{dp_i/obj})_{z} = \text{twisting}$$

The notations $()_x$, $()_y$, $()_z$, $()_{x,y}$ and so on stand of course for the corresponding coordinates of the vector they enclose, z being the normal in the case of the contact frame.

The assumption of non-sliding and the condition of non-breaking combine into:

$$v_{c_i \in dp_i/obj} = 0_{3,1}$$
$$\text{in other words: } \Pi V^{c_i}_{dp_i/obj} = 0_{3,1}$$

Free rolling and free twisting imply that no moment can be applied by the finger on the object at the contact point:

$$m_{c_i, dp_i \to obj} = 0_{3,1}$$
$$\text{in other words: } \Pi' W^{c_i}_{dp_i \to obj} = 0_{3,1}$$

III. Modeling equations

A. Basic equations

In this work, we will not use joint stiffnesses, that is to say we will remain at cartesian phalanx-level and object-level. The stiffness mappings at these levels are pictured in figure 2, equations (1) and (2).

In this figure, (1) and (2) are stiffness definitions, (3) is the differentiation of $W_{dp \to obj} = \sum_{i=1}^{n_f} W^{obj}_{dp_i \to obj}$ after the change of frame $W^{obj}_{dp_i \to obj} = {}^{obj}Ad^{-T}_{dp_i} W^{dp_i}_{dp_i \to obj}$, (4) is a mere velocity-addition law and (5) is the assumption of non-sliding combined with the condition of non-breaking, see section II-C above.

As explained in the introduction, we are interested in the relation between K_{obj} and K_{dp_i}. Namely, we look for K_{obj} as a function of the different K_{dp_i}.

In order to get this relation (1), we will need no more than six modeling equations: the basic equations (2) to (5), the assumption of free rolling and free twisting at contact in section II-C, and a kinematic equation (9) derived in section III-B from Montana's kinematic equations of contact [6].

These modeling equations will enable us to find a linear relation between $dW_{dp \to obj}$ and δX_{obj}, therefore proving constructively the existence of a stiffness relation (1) in cartesian space, at object level. At the same time, we will get K_{obj} as a function of the different K_{dp_i}.

B. A kinematic equation of pure rolling contact

In order to describe the motion of the contact point c_i on the phalanx dp_i, we define the following twist:

$$V^{c_i}_{c_i/dp_i} = \begin{pmatrix} v^{c_i}_{c_i/dp_i} \\ \omega^{c_i}_{c_i/dp_i} \end{pmatrix}$$

At first glance, one could think that such a twist makes little sense: the c_i in c_i/dp_i is supposed to be a rigid body, but there is no such rigid body at the interface between the finger and the object; besides, the linear velocity should be $v^{c_i}_{c_i \in c_i/dp_i}$, which is hardly intelligible. In fact, the c_i in c_i/dp_i means here a mere virtual rigid body to which the frame c_i is rigidly linked (hence the same notation). Therefore, the first c_i in $v^{c_i}_{c_i \in c_i/dp_i}$ is the contact point, the second is the virtual body and the third is the frame of expression: this velocity is indeed the velocity of the contact point c_i in its motion on the distal phalanx dp_i, written in the basis c_i, and we rather note it $v^{c_i}_{c_i/dp_i}$. Similarly, $\omega^{c_i}_{c_i/dp_i}$ is the rotational velocity of the contact frame c_i relatively to dp_i, written in the basis c_i.

This being clear, it is possible to translate into our notations the kinematic equations of contact proven by [6]. These equations are a system of identities that relate the velocities of the contact point on the distal phalanx (v_{c_i/dp_i}) and on the object (would be $v_{c_i/obj}$, however we do not use it) with the translational and rotational velocities of the relative motion between the phalanx and the object (velocities that we have noted $v_{c_i \in dp_i/obj}$ and $\omega_{dp_i/obj}$, and of which different components are the sliding, rolling, twisting and breaking velocities). These relations between the motion of the contact point across the surfaces in contact and the relative motion of the surfaces are functions of the *geometric parameters* of the surfaces only, namely, their metrics, curvature forms and torsion forms.

As we cannot re-expose all the concepts from differential geometry that are necessary to the total understanding of the kinematic equations of contact, we refer the reader to [6], or any subsequent reference book on multi-fingered manipulation that deals with the kinematics of rolling contacts, for instance [7] or [8].

In our notations, the first kinematic equation of contact (and the only one we will need) reads [6, equation 17]:

$$(v^{c_i}_{c_i/dp_i})_{x,y} = (\Gamma^{c_i}_{dp_i} + \Gamma^{c_i}_{obj})^{-1} \dots$$
$$\dots \left[\begin{pmatrix} -(\omega^{c_i}_{dp_i/obj})_y \\ (\omega^{c_i}_{dp_i/obj})_x \end{pmatrix} - \Gamma^{c_i}_{obj} \begin{pmatrix} (v^{c_i}_{c_i \in dp_i/obj})_x \\ (v^{c_i}_{c_i \in dp_i/obj})_y \end{pmatrix} \right] \quad (6)$$

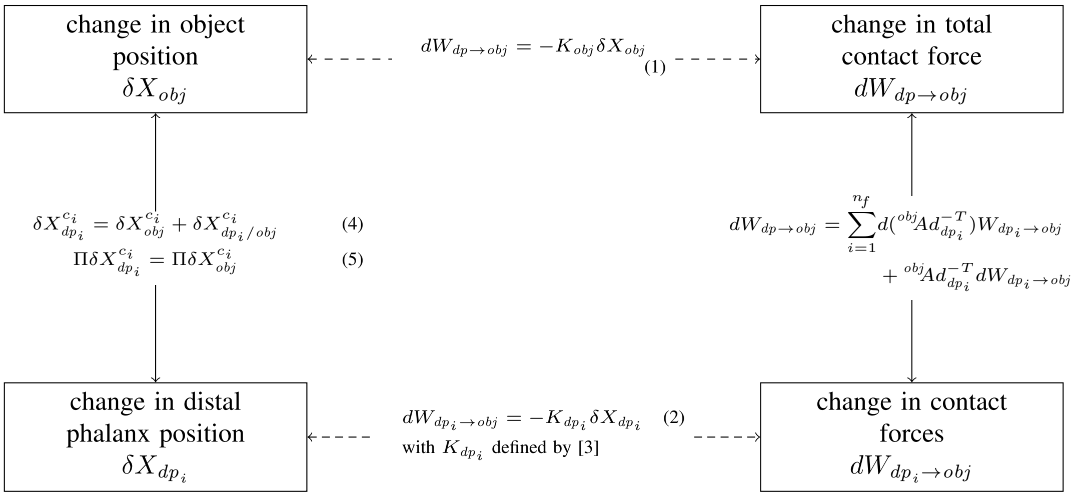


Fig. 2. Stiffness mappings in cartesian space, at phalanx and object levels

In this equation, $\Gamma_{dp_i}^{c_i}$ and $\Gamma_{obj}^{c_i}$ denote (2,2) matrices that are the curvature forms of the surfaces, at the point of contact and relatively to the x and y axes of the contact frame c_i.

The original formulation of (6) also involves the metric tensor of the phalanx surface, however this tensor is a function of the local parameterization chosen for the surface around the contact point. In our case, we can choose at each time t a convenient, orthonormal local coordinate chart to parameterize the phalanx surface around the contact point. This yields a metric tensor equal to the identity matrix I_2.

Because of non-sliding, (6) may be simplified as:

$$(v_{c_i/dp_i}^{c_i})_{x,y} = (\Gamma_{dp_i}^{c_i} + \Gamma_{obj}^{c_i})^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(\omega_{dp_i/obj}^{c_i})_{x,y} \quad (7)$$

Since the contact point always remains on the surface of the phalanx (!), we have $(v_{c_i/dp_i}^{c_i})_z = 0$, an identity that enables the rewriting of (7) as:

$$\begin{aligned} v_{c_i/dp_i}^{c_i} &= \begin{pmatrix}(\Gamma_{dp_i}^{c_i} + \Gamma_{obj}^{c_i})^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & 0_{2,1}\\ 0_{1,2} & 0\end{pmatrix}\omega_{dp_i/obj}^{c_i} \\ &\overset{\text{def}}{=} \widehat{\Gamma}_{dp_i,obj}\,\omega_{dp_i/obj}^{c_i} \end{aligned} \quad (8)$$

Using the matrices Π and Π' defined in section II-A, (8) may be rewritten as:

$$\Pi V_{c_i/dp_i}^{c_i} = \widehat{\Gamma}_{dp_i,obj}\Pi' V_{dp_i/obj}^{c_i}$$

Last, we multiply by dt and use the velocity-addition law (4):

$$\Pi\delta X_{c_i/dp_i}^{c_i} = \widehat{\Gamma}_{dp_i,obj}\Pi'(\delta X_{dp_i}^{c_i} - \delta X_{obj}^{c_i}) \quad (9)$$

C. Two identities about the contact forces

An obvious result of free rolling and free twisting is (see section II-C):

$$\Pi' dW_{dp_i\to obj}^{c_i} = 0_{3,1} \quad (10)$$

Another consequence is that the contact wrench at contact i reads, in matrix form and in c_i coordinates:

$$\widehat{W}_{dp_i\to obj}^{c_i} = \begin{pmatrix}0_{3,3} & \hat{f}_{dp_i\to obj}^{c_i}\\ \hat{f}_{dp_i\to obj}^{c_i} & 0_{3,3}\end{pmatrix}$$

and thanks to this specific, anti-diagonal form, it meets the following identity (easy to verify):

$$\Pi'\widehat{W}_{dp_i\to obj}^{c_i} = \Pi'\widehat{W}_{dp_i\to obj}^{c_i}\Pi^T\Pi \quad (11)$$

IV. The linear map between δX_{obj} and δX_{dp_i}

Now we use the modeling equations enumerated in section III to prove the first contribution of this paper: the infinitesimal pose variations of the object and a distal phalanx, δX_{obj} and δX_{dp_i}, are linearly dependent one another. We provide an expression of this linear map, in the coordinates of the contact frame, that is to say in terms of $\delta X_{obj}^{c_i}$ and $\delta X_{dp_i}^{c_i}$ (only appropriate adjoints are needed to write this linear relation in other coordinates).

A. An expression of $dW_{dp_i\to obj}^{c_i}$

$W_{dp_i\to obj}$ is the contact wrench applied by finger i on the object; its expressions in the contact frame c_i and in the distal phalanx frame dp_i are related through the following change of frame:

$$W_{dp_i\to obj}^{c_i} = {}^{c_i}Ad_{dp_i}^{-T}W_{dp_i\to obj}$$

Differentiating this relation and using the definition of phalanx-level stiffness (2), as well as a simple change of frame, yield:

$$\begin{aligned} dW_{dp_i\to obj}^{c_i} &= d({}^{c_i}Ad_{dp_i}^{-T})W_{dp_i\to obj} \\ &\quad - {}^{c_i}Ad_{dp_i}^{-T}K_{dp_i}{}^{dp_i}Ad_{c_i}\delta X_{dp_i}^{c_i} \end{aligned} \quad (12)$$

Then we use the property (23) proven in the appendix to rewrite (12) as:

$$\begin{aligned} dW_{dp_i\to obj}^{c_i} &= \widehat{W}_{dp_i\to obj}^{c_i}\delta X_{c_i/dp_i}^{c_i} \\ &\quad - {}^{c_i}Ad_{dp_i}^{-T}K_{dp_i}{}^{dp_i}Ad_{c_i}\delta X_{dp_i}^{c_i} \end{aligned} \quad (13)$$

It is worth understanding that in the notation $\delta X_{c_i/dp_i}^{c_i}$, according to (23), the c_i in c_i/dp_i is a rigid body to which the frame c_i is rigidly linked. As we said previously, there is no such rigid body except a virtual one: the $\delta X_{c_i/dp_i}^{c_i}$ coming from the application of (23) is exactly the one we used in section III-B. Consequently, we are entitled to use (9), proven in this section. We will also use the previous developments (10) and (11).

B. A first relation between $\delta X^{c_i}_{dp_i}$ and $\delta X^{c_i}_{obj}$

First we pre-multiply (13) by Π' and use (10) to write:

$$\begin{aligned} 0_{3,1} &= \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \delta X^{c_i}_{c_i/dp_i} \\ &\quad - \Pi' {}^{c_i}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} \delta X^{c_i}_{dp_i} \end{aligned}$$

Thanks to (11), this equation becomes:

$$\begin{aligned} 0_{3,1} &= \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \Pi \delta X^{c_i}_{c_i/dp_i} \\ &\quad - \Pi' {}^{c_i}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} \delta X^{c_i}_{dp_i} \end{aligned}$$

Then (9) yields:

$$\begin{aligned} 0_{3,1} &= \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' (\delta X^{c_i}_{dp_i} - \delta X^{c_i}_{obj}) \\ &\quad - \Pi' {}^{c_i}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} \delta X^{c_i}_{dp_i} \end{aligned}$$

Eventually we group the resulting terms:

$$\begin{aligned} 0_{3,1} &= (\Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' - \ldots \\ &\qquad \ldots \Pi' {}^{c_i}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i}) \delta X^{c_i}_{dp_i} \\ &\quad - \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' \delta X^{c_i}_{obj} \end{aligned} \tag{14}$$

C. A second relation between $\delta X^{c_i}_{dp_i}$ and $\delta X^{c_i}_{obj}$

Equation (14) is a system of three scalar linear equations relating the (6,1) vectors $\delta X^{c_i}_{dp_i}$ and $\delta X^{c_i}_{obj}$: it is not sufficient to derive the one as a linear function of the other.

However, $\delta X^{c_i}_{dp_i}$ and $\delta X^{c_i}_{obj}$ are also related through the equation of non-sliding (5), that provides three other scalar linear equations:

$$\Pi \delta X^{c_i}_{dp_i} = \Pi \delta X^{c_i}_{obj} \tag{15}$$

D. Conclusion

Eventually, with (14) and (15) we have a system of six scalar linear equations, (16).

Let us define the following matrices:

$$\Xi_{dp_i} = \begin{pmatrix} \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' - \Pi' {}^{c_i}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} \\ \Pi \end{pmatrix}$$

$$\Xi_{obj,i} = \begin{pmatrix} \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' \\ \Pi \end{pmatrix}$$

Equation (16) may be rewritten:

$$\Xi_{dp_i} \delta X^{c_i}_{dp_i} - \Xi_{obj,i} \delta X^{c_i}_{obj} = 0_{6,1} \tag{17}$$

To solve this system in the variables $\delta X^{c_i}_{dp_i}$, we assume that the matrix Ξ_{dp_i} is invertible. We get the following expression of $\delta X^{c_i}_{dp_i}$ as a linear function of $\delta X^{c_i}_{obj}$, which is what we were looking for in this section:

$$\delta X^{c_i}_{dp_i} = \Xi^{-1}_{dp_i} \Xi_{obj,i} \delta X^{c_i}_{obj} \tag{18}$$

V. Expression of K_{obj} as a function of K_{dp_i}

In this section, we use the basic modeling equations of section III-A and the previous result (18) to prove the main contribution of this paper: an expression of the cartesian object-level stiffness matrix K_{obj} as a function of finger cartesian stiffness matrices K_{dp_i}.

A. An expression of $dW_{dp \to obj}$

First we use (2), (3), and a simple change of frame to get the following expression of $dW_{dp \to obj}$:

$$\begin{aligned} dW_{dp \to obj} &= \sum_{i=1}^{n_f} d({}^{obj}Ad^{-T}_{dp_i}) W_{dp_i \to obj} \\ &\qquad - {}^{obj}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} \delta X^{c_i}_{dp_i} \end{aligned} \tag{19}$$

Then we use successively the property (23) proven in the appendix, a change of frame and the velocity-addition law (4) to rewrite the first term of the right-hand side of (19) as:

$$\begin{aligned} d({}^{obj}Ad^{-T}_{dp_i}) W_{dp_i \to obj} &= \widehat{W}^{obj}_{dp_i \to obj} \delta X^{obj}_{obj/dp_i} \\ &= \widehat{W}^{obj}_{dp_i \to obj} {}^{obj}Ad_{c_i} \delta X^{c_i}_{obj/dp_i} \\ &= \widehat{W}^{obj}_{dp_i \to obj} {}^{obj}Ad_{c_i} (\delta X^{c_i}_{obj} - \delta X^{c_i}_{dp_i}) \end{aligned}$$

Consequently, (19) depends linearly on $\delta X^{c_i}_{obj}$ and $\delta X^{c_i}_{dp_i}$:

$$\begin{aligned} dW_{dp \to obj} &= \sum_{i=1}^{n_f} \widehat{W}^{obj}_{dp_i \to obj} {}^{obj}Ad_{c_i} \delta X^{c_i}_{obj} \\ &\quad - ({}^{obj}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} + \widehat{W}^{obj}_{dp_i \to obj} {}^{obj}Ad_{c_i}) \delta X^{c_i}_{dp_i} \end{aligned} \tag{20}$$

B. Conclusion

We replace (18) into (20) and get (21). This proves constructively the existence of the stiffness relation (1) in cartesian space, at object level (under the hypotheses of our model and the assumption of invertibility of Ξ_{dp_i}). Eventually, we also get K_{obj} as a function of K_{dp_i} (22).

C. Remarks

We can see in the expression of K_{obj} that it embeds a variety of contributions:

- The stiffnesses of the fingers of course, through the various K_{dp_i} (also present in Ξ_{dp_i}). These cartesian stiffness matrices themselves embed the stiffnesses of the joints and the contributions to stiffness of the changes in the geometry of the fingers as they move under the effect of the contact forces [2], [3].
- The contact forces contribute a second time to K_{obj} through the $\widehat{W}_{dp_i \to obj}$ (also present in Ξ_{dp_i} and $\Xi_{obj,i}$).
- The relative curvatures, at the contact points, of the surfaces of the fingers and object, contribute through the terms $\widehat{\Gamma}_{dp_i,obj}$ (also present in Ξ_{dp_i} and $\Xi_{obj,i}$).
- A number of lever arms, involved in transposing the effects of the contact forces and finger cartesian stiffnesses, from the surface of the object or from the center of the phalanxes to the center of mass of the object, also contribute to K_{obj} through the various co-adjoint matrices.

$$\begin{cases} (\Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' - \Pi' {}^{c_i}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i}) \delta X^{c_i}_{dp_i} - \Pi' \widehat{W}^{c_i}_{dp_i \to obj} \Pi^T \widehat{\Gamma}_{dp_i,obj} \Pi' \delta X^{c_i}_{obj} = 0_{3,1} \\ \Pi \delta X^{c_i}_{dp_i} - \Pi \delta X^{c_i}_{obj} = 0_{3,1} \end{cases} \tag{16}$$

$$dW_{dp \to obj} = \sum_{i=1}^{n_f} \left[\widehat{W}^{obj}_{dp_i \to obj} - ({}^{obj}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} + \widehat{W}^{obj}_{dp_i \to obj} {}^{obj}Ad_{c_i}) \Xi^{-1}_{dp_i} \Xi_{obj,i} {}^{c_i}Ad_{obj} \right] \delta X_{obj} \tag{21}$$

$$K_{obj} = \sum_{i=1}^{n_f} \left[({}^{obj}Ad^{-T}_{dp_i} K_{dp_i} {}^{dp_i}Ad_{c_i} + \widehat{W}^{obj}_{dp_i \to obj} {}^{obj}Ad_{c_i}) \Xi^{-1}_{dp_i} \Xi_{obj,i} {}^{c_i}Ad_{obj} - \widehat{W}^{obj}_{dp_i \to obj} \right] \tag{22}$$

Stiffness matrices in robotics are usually defined as symmetric, positive definite matrices, or at least positive semi-definite. It is not obvious from the expression (22) whether it is the case for K_{obj}. Besides, there is a case for generaly asymmetric stiffness matrices, only a submatrix of which would be positive (semi)-definite:

- Asymmetric cartesian stiffness matrices were introduced and discussed during the 1990s by various researchers, in particular Griffis, Duffy and Pigoski [9], [10], Ciblak and Lipkin [11], and Žefran, Kumar and Howard [12]–[14]. In turn, Chen, Li and Kao exposed in a series of papers why the cartesian stiffness matrix yielded by their conservative congruence transformation is not symmetric in general [15]–[21].
  The previous works concluded that in general, a cartesian $(6, 6)$ stiffness matrix at end-effector level is asymmetric. It becomes symmetric when the manipulator is unloaded; or when the twists, and consequently the stiffness matrix, are expressed in a *coordinate* basis of the twist space, rather than in the usual *non-coordinate* basis (consisting of three translational velocities and three rotational velocities around the same axes); or when it is restricted to its $(3, 3)$ translational part.
  In our modeling, we used the usual, non-coordinate basis of the twist space, and there is a load at the end-effectors of the fingers. As a result the matrices K_{dp_i} are not expected to be symmetric. Should the resulting K_{obj} be always symmetric then, it would be surprising.
- Depending on the finger structure (number of degrees of freedom and how their axes are arranged) and on the grasp geometry, there may be cases where the distal phalanxes cannot move in the six directions of the twist space. A straightforward example is a planar finger: its distal phalanx has three blocked directions, one in translation and two in rotation. Such blocked directions are directions of infinite stiffness, and the corresponding terms in the cartesian K_{dp_i} would be $+\infty$. Likewise, a planar two-finger pinch grasp would have a resulting K_{obj} with the same blocked directions.
  As a result, neither K_{obj} nor K_{dp_i} would qualify as stiffness matrices in the canonical sense of a symmetric, positive (semi)-definite matrix. Yet it remains possible that adequate submatrices correctly describe the elastic behavior of the grasp realized by the stiff fingers.
  We should however note that the assumptions of free rolling and free twisting we made directly limit the happening of cases like the one we are speaking about. For such cases outside our model hypotheses, the expression of K_{obj}, or of what would be in case a stiffness relation does not exist, is still to find.

From these remarks, it appears that there is still a lot of work ahead to investigate completely the structure, properties and physical meaning of the cartesian object-level stiffness.

VI. Numerical insights

As a first numerical test of our modeling, we designed a simple simulated experiment with ARBORIS [22], the dynamical engine we used in our previous works [23], [24].

This test involves a spherical object of radius $r_{obj} = 2$ cm grasped by a tetrahedron grasp of four "cartesian" fingers, namely spheres of radius $r_{dp} = 5$ mm and cartesian stiffnesses $K_{dp_i} = \text{diag}(k_{dp,tr} I_3, k_{dp,rot} I_3)$, for various values of $k_{dp,tr}$ and $k_{dp,rot}$. The object is subject to small displacements in the six directions of space (1 mm for the three translations, 10 ° for the three rotations), and we compare the resulting $dW_{dp \to obj}$ in the simulation with the one from (21). The results are summarized in figure 3.

It appears that the $dW_{dp \to obj}$ predicted by (21), i.e. by our modeling of K_{obj}, correctly match the simulated ones.

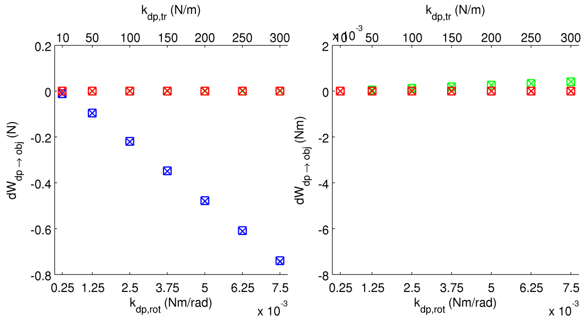
VII. Conclusion

A. Summary

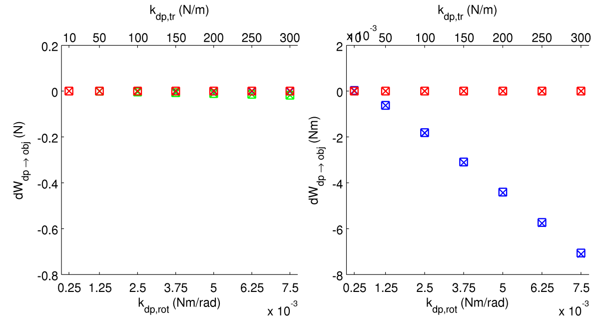
In this paper, we demonstrated an expression of the cartesian matrix that models the behavior of an object grasped by a multi-fingered robot hand with stiff joints. We showed that this expression is a non-linear function of the finger cartesian stiffness matrices and depends also on the contact forces and local geometries of the contacting surfaces. The result we propose is valid under the assumptions that the phalanxes and the object are rigid bodies, that the contacts are non-sliding, non-breaking point contacts with free rolling and free twisting, and that a certain matrix encountered during the modeling is invertible.

B. Future work

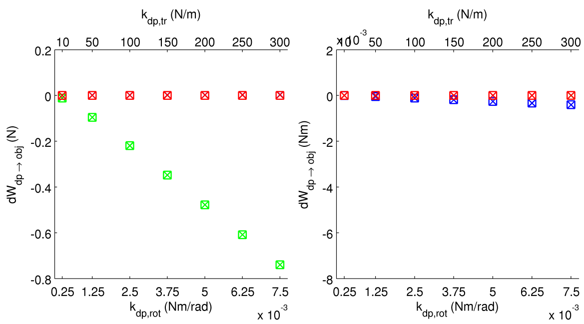
We already underlined in section V-C the work ahead in the understanding of the structure, properties and physical meaning of the cartesian object-level stiffness matrix. In particular, we should investigate what happens when model hypotheses are removed or at least restricted. The motivation



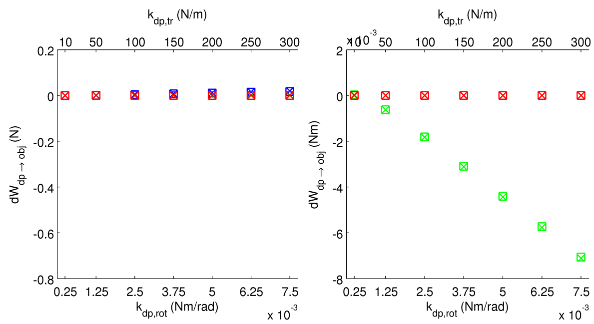
(a) $\delta X_{obj} = 1$ mm along x



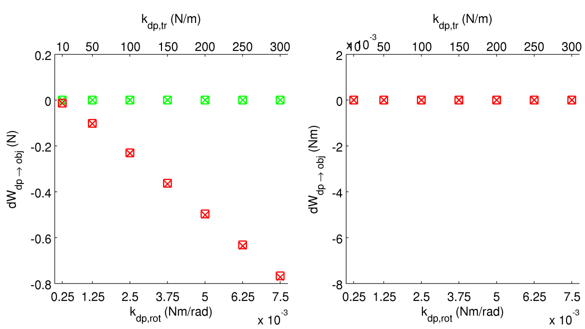
(b) $\delta X_{obj} = 10\,^\circ$ around x



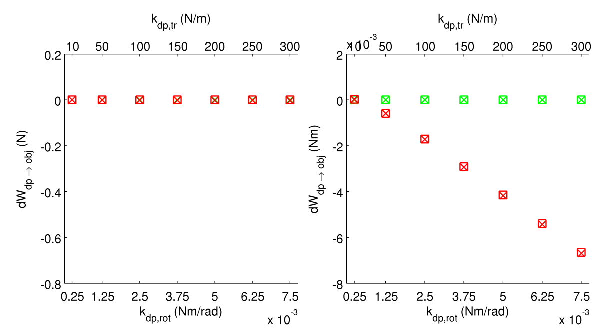
(c) $\delta X_{obj} = 1$ mm along y



(d) $\delta X_{obj} = 10\,^\circ$ around y



(e) $\delta X_{obj} = 1$ mm along z



(f) $\delta X_{obj} = 10\,^\circ$ around z

Fig. 3. $dW_{dp \to obj}$ such that predicted by (21) (boxes) and returned by the simulation (ticks), for various values of $k_{dp,tr}$ (top horizontal axis) and $k_{dp,rot}$ (bottom horizontal axis). We tried six different cases of δX_{obj}; in each case, the left subplot is the force part of $dW_{dp \to obj}$ (in N) and the right subplot is its moment part (in Nm). Blue, green, red correspond respectively to the x, y, z coordinates of these force and moment parts, in the object basis (blue boxes and ticks are sometimes hidden by the green or red ones at the zero horizontal line). Ticks and boxes coincide, meaning that the values for $dW_{dp \to obj}$ predicted by (21), i.e. by our modeling of K_{obj}, correctly match the simulated (experimental) ones.

for that is that we all know of fingers whose distal phalanxes have indeed limited, if not blocked, directions of motion, and whose grasps are still very able to produce an object-level stiffness without any blocked direction: our very own fingers.

The invertibility of Ξ_{dp_i} is also an issue to study. More numerical simulations, if not actual experiments, should be done to validate or challenge our modeling. In particular, stiffness control of a multi-fingered grasp, based on the expression we propose for K_{obj}, should be tried out.

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APPENDIX

In this appendix, we prove the following property, used twice in this document: let S_1, S_2, S_a and S_b denote four rigid bodies and a and b denote two frames rigidly linked to S_a and S_b respectively. Then:

$$\frac{d}{dt}({}^a\!Ad_b^{-T})W^b_{S_2\to S_1} = \widehat{W}^a_{S_2\to S_1}V^a_{S_a/S_b} \tag{23}$$

The proof we propose here relies on three preliminary results from rigid body mechanics, that we recall in the next three sections, with only outlines of their demonstrations because of place constraints.

A. First lemma: change of frame of a twist-relative cross-product matrix

First of all, we recall the change of basis formula for a (3,3) cross-product matrix: $\hat{r}^a = {}^aR_b\hat{r}^b\,{}^bR_a$. There is a similar result about the change of frame formula for a twist-relative cross-product matrix:

$$\widehat{V}^a_{S_2/S_1} = {}^a\!Ad_b\widehat{V}^b_{S_2/S_1}\,{}^b\!Ad_a \tag{24}$$

The proof is elementary, though not straightforward. We start by calculating the right-hand side of (24). After a number of changes of bases for various cross-product matrices:

$$\begin{aligned}&{}^a\!Ad_b\widehat{V}^b_{S_2/S_1}\,{}^b\!Ad_a = \dots\\ &\begin{pmatrix}\hat{\omega}^a_{S_2/S_1} & \hat{v}^a_{B\in S_2/S_1} + \hat{r}^a_{a,b}\hat{\omega}^a_{S_2/S_1} - \hat{\omega}^a_{S_2/S_1}\hat{r}^a_{a,b}\\ 0_{3,3} & \hat{\omega}^a_{S_2/S_1}\end{pmatrix}\end{aligned}$$

From Jacobi identity it is possible to prove $\hat{r}^a_{a,b}\hat{\omega}^a_{S_2/S_1} - \hat{\omega}^a_{S_2/S_1}\hat{r}^a_{a,b} = (r^a_{a,b}\times\omega^a_{S_2/S_1})^\frown$. Indeed, right-multiplying this later identity by some vector u yields $r^a_{a,b}\times(\omega^a_{S_2/S_1}\times u^a) + \omega^a_{S_2/S_1}\times(u^a\times r^a_{a,b}) = (r^a_{a,b}\times\omega^a_{S_2/S_1})\times u^a$ in a few rewritings. Consequently, the top-right term in the previous matrix may be rewriten $(v^a_{B\in S_2/S_1} + r^a_{a,b}\times\omega^a_{S_2/S_1})^\frown = \hat{v}^a_{A\in S_2/S_1}$.

B. Second lemma: a remarkable identity

We have the following remarkable identity:

$$(\widehat{V}^a_{S_2/S_1})^TW^a_{S_3\to S_4} + (\widehat{W}^a_{S_3\to S_4})^TV^a_{S_2/S_1} = 0_{6,1} \tag{25}$$

The proof consists in trivial matrix calculus and using the skew-symmetry of $\widehat{W}_{S_3\to S_4}$.

C. Third lemma: time derivative of an adjoint or co-adjoint matrix

In this section, contrary to the two previous ones, the rigid bodies S_1 and S_2 are specific: they are rigidly linked to the frames a and b respectively, so we denote them S_a and S_b.

The time derivative of the adjoint matrix ${}^a\!Ad_b$ has the following expression:

$$\frac{d}{dt}{}^a\!Ad_b = {}^a\!Ad_b\widehat{V}^b_{S_b/S_a}$$

The proof uses the fact that $\hat{v}^b_{B\in S_b/S_a} = {}^bR_a\hat{v}^a_{B\in S_b/S_a}\,{}^aR_b = {}^bR_a\hat{\dot{r}}^a_{a,b}\,{}^aR_b$ and $\hat{\omega}^b_{S_b/S_a} = {}^bR_a\,{}^a\dot{R}_b$ to decompose $\frac{d}{dt}{}^a\!Ad_b$ into the product of ${}^a\!Ad_b$ and $\widehat{V}^b_{S_b/S_a}$.

It is possible to prove similarly the following expression of the time derivative of the inverse adjoint matrix ${}^a\!Ad_b^{-1}$:

$$\frac{d}{dt}({}^a\!Ad_b^{-1}) = -\widehat{V}^b_{S_b/S_a}\,{}^a\!Ad_b^{-1} \tag{26}$$

D. Conclusion: proof of the property

First we transpose (26) and get:

$$\frac{d}{dt}({}^a\!Ad_b^{-T}) = -{}^a\!Ad_b^{-T}(\widehat{V}^b_{S_b/S_a})^T$$

Then we use (24) to rewrite this equation as:

$$\begin{aligned}d({}^a\!Ad_b^{-T}) &= -{}^a\!Ad_b^{-T}({}^b\!Ad_a\widehat{V}^a_{S_b/S_a}\,{}^a\!Ad_b)^Tdt\\ &= -(\widehat{V}^a_{S_b/S_a})^T\,{}^a\!Ad_b^{-T}dt\end{aligned}$$

From this identity and the lemma (25), we deduce:

$$\begin{aligned}d({}^a\!Ad_b^{-T})W^b_{S_2\to S_1} &= -(\widehat{V}^a_{S_b/S_a})^TW^a_{S_2\to S_1}dt\\ &= (\widehat{W}^a_{S_2\to S_1})^TV^a_{S_b/S_a}dt\end{aligned}$$

And as $\widehat{W}_{S_2\to S_1}$ is skew-symmetric:

$$\begin{aligned}d({}^a\!Ad_b^{-T})W^b_{S_2\to S_1} &= -\widehat{W}^a_{S_2\to S_1}V^a_{S_b/S_a}dt\\ &= \widehat{W}^a_{S_2\to S_1}V^a_{S_a/S_b}dt\end{aligned}$$

This last identity is (23).